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EQUATION OF MOTION OF AN EXPANDING VAPOUR DROP IN AN IMMISCIBLE LIQUID MEDIUM

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Abstract—Using the description suggested by Sideman for the internal evaporation of a drop with the growth of a liquid sheath suspended from the lower part of the expanding bubble, an equation describing the motion of such a system is derived. The equation is solved numerically, and the results are compared both with experimental data given in [2] and the authors' own experimental results. There is good agreement between experiment and theory.

NOMENCLATURE

B,	dimensionless constant;
C_D,	coefficient of frontal resistance, equation (4);
C,	constant, equation (7a);
C_p,	specific heat of evaporating liquid $\left[\frac{\text{cal}}{\text{g deg}}\right]$;
g,	acceleration of gravity $\left[\frac{\text{cm}}{\text{s}^2}\right]$;
H,	actual position of system [cm];
H_0,	initial position of system [cm];
K,	constant in equation (12);
K_c,	coefficient of equation (3);
L,	constant of equation (12);
L_p,	heat of vaporization $\left[\frac{\text{cal}}{\text{g}}\right]$;
M,	constant of equation (12);
M_{cz},	molecular mass,
N_{Ja},	Jacob's number [dimensionless];
P,	tension of saturated vapour $\left[\frac{\text{dyn}}{\text{cm}^2}\right]$;
p,	dimensionless constant;
p_w,	internal pressure of bubble $\left[\frac{\text{dyn}}{\text{cm}^2}\right]$;
R,	bubble radius [cm];
R^x,	effective radius [cm];
T,	temperature [°K];
t_0,	starting time of system [s];
V,	velocity of system $\left[\frac{\text{cm}}{\text{s}}\right]$;
V_0,	initial volume of drop [cm^3].

Greek symbols

α,	thermal diffusivity $\left[\frac{\text{cm}^2}{\text{s}}\right]$;
β,	opening angle [rad];
φ,	coefficient;
σ,	surface tension $\left[\frac{\text{dyn}}{\text{cm}}\right]$;
ρ_L,	density of continuous phase [g/cm^3];
ρ_{L_p},	density of evaporating liquid [g/cm^3];
ρ_v,	vapour density [g/cm^3].

INTRODUCTION

IN OUR previous work [1] dealing with the mechanism of evaporation of a liquid drop immersed in a superheated (with respect to its boiling point) immiscible liquid medium, we pointed out that depending on the ratio of surface tensions of both liquids there must be at least two completely different mechanisms of evaporation.

The first is connected with the evaporation of liquid into a drop which thus takes the shape of a flat sheath suspended from the expanding vapour bubble, and the other is connected with the removal of vapour nuclei from the surface of the superheated drop to the surrounding liquid. Different mechanisms exhibit different degrees of effectiveness of membraneless heat exchange. Explanation of the functioning of these mechanisms might have considerable theoretical and practical importance. The mechanism of internal evaporation of a drop with the growth of a liquid sheath suspended from an expanding vapour bubble has been described by Sideman *et al.* in several works ([2, 3] among others).

Basing on the results of his experimental studies of the motion of expanding bubbles, Sideman derived the following empirical formula for the water–pentane

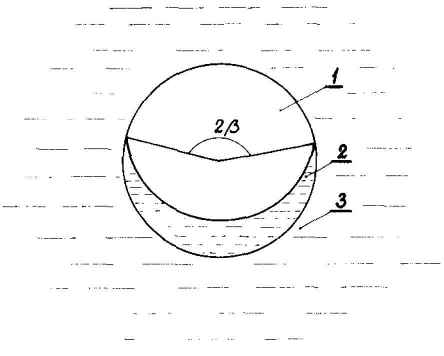

FIG. 1. Expanding vapour drop: (1) vapour; (2) evaporating liquid; (3) continued phase.

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system describing the relationship between position of the bubble and time:

$$H = H_0 + B \cdot t^p \tag{1}$$

where H is the position of bubble after time t, H_0, the position at the time instant t_0 (separation from the capillary), t, time, and B, p are constants.

MATHEMATICAL DESCRIPTION

In this work we attempt to derive analytically the equation of motion of a system composed of an expanding bubble with a suspended sheath of evaporating liquid. In order to do this we consider the system as moving in a uniformly superheated liquid. In accordance with experimental observations, we assume that the upper part of the system while going upwards has a spherical shape, and the loading of the suspended liquid up to about 90% of evaporation prevents the lower part of the system from being deformed to the shape of a "mushroom".

The momentum balance of the moving system:

$$\boxed{\begin{array}{l}\text{change of}\\ \text{momentum of}\\ \text{moving system}\end{array}} = \boxed{\begin{array}{l}\text{upthrust}\\ \text{on the}\\ \text{bubble}\end{array}} - \boxed{\begin{array}{l}\text{resistance}\\ \text{of medium}\\ \end{array}} - \boxed{\begin{array}{l}\text{apparent weight}\\ \text{of suspended}\\ \text{liquid}\end{array}}$$

The momentum of the moving system is made up of the following components: $\rho_v \frac{4}{3}\pi R^3 V$, momentum of vapour bubble; $m(t) \cdot V$, momentum of suspended liquid; $\frac{1}{2}\rho_L \frac{4}{3}\pi R^3 V$, momentum of external liquid forced upwards with the bubble [12].

From the above balance we get the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_v \tfrac{4}{3}\pi R^3 V + m(t) V + \tfrac{1}{2}\rho_L \tfrac{4}{3}\pi R^3 V\right)$$
$$= (\rho_L - \rho_v)\tfrac{4}{3}\pi R^3 g - C_D \pi R^2 \frac{\rho_L \cdot V^2}{2} - g \cdot (\rho_{L_p} - \rho_L)$$
$$\times \left(V_0 - \frac{2\pi(1+\cos\beta)\varphi R^2 t}{\rho_{L_p}}\right) \tag{2}$$

where $m(t)$ is the mass of suspended liquid, and β is the "opening angle" of vapour phase defined in Fig. 1.

The coefficient φ in the last part of the equation determines the quantity of evaporating liquid from unit surface in unit time for given pressure and temperature. From the molecular-kinetic theory of gases [4], for moderate values of external pressure (approaching atmospheric pressure) the coefficient φ can be determined from the following relationship:

$$\varphi = K_c \cdot M_{cz} \cdot P \cdot \left(\frac{3}{MRT}\right)^{1/2} \tag{3}$$

where M_{cz} is the molecular mass of evaporating liquid and P, the vapour pressure of saturated evaporating liquid.

Saturated vapour pressure P in the expression (3) requires further discussion. This pressure is not constant in time. It depends on the surface curvature of the sheath, that is on the bubble radius and on the hydrostatic pressure, that is on the position of the bubble. The average value of this pressure, however, does not differ from the extreme values by more than 10%, which gives us the right to assume that over this range of accuracy the average value of pressure is constant.

The coefficient of resistance C_D in the last but one element of the equation is defined as:

$$C_D = \frac{\text{resistance}}{\text{frontal surface} \cdot (V^2/2\mathrm{g})}. \tag{4}$$

As shown in [5], the coefficient C_D is a complex function of a number of parameters. Some cases of motion of systems with regular shapes have been solved. For large bubbles ($R \geqslant 0.07$ cm) whose shape in nearly spherical, it is most convenient to determine the coefficient of resistance from the relationship given in [6, 7], that is:

$$C_D = \tfrac{8}{3}\frac{\rho_L \cdot g}{1.82\sigma} R^2 \tag{5}$$

where σ is the surface tension at the bubble-surrounding liquid boundary.

In order to solve equation (2) the rate of change of the bubble radius should also be determined.

The temperature conditions in the sheath of the evaporating liquid can be regarded as fixed. One can, therefore, describe the growth of this bubble by means of expressions analogous to Rayleigh's equation related to bubble growth by evaporation of suspended liquid surrounding the bubble.

The temperature at the internal surface of the liquid is equal to the temperature of saturation of the evaporating liquid, a heat flux of constant density is flowing from the surrounding liquid to the liquid sheath.

Rayleigh's equation describing the growth of the bubble when evaporation occurs over the whole internal surface takes the following form:

$$\rho_{L_p}\left[R \cdot \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} + \tfrac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right] = p_w - \frac{2\sigma_1}{R} \tag{6}$$

where σ_1 is the surface tension at the evaporating liquid–vapour boundary.

From the solutions of this equation presented in [8–10] it follows that the form of the function $R(t)$ can be assumed to be as follows:

$$R = 2N_{Ja}(3\alpha t/\pi)^{1/2} \tag{7}$$

where

$$N_{Ja} = \rho_{L_p} C_{L_p} (T_\infty - T_s) \rho_v \cdot L$$

T_∞ being the internal temperature of superheated liquid and T_s the temperature of saturation.

Some authors, depending on the assumptions taken, give solutions somewhat different from the form (7). Nonetheless the relationship always takes the following form:

$$R = C \cdot t^{1/2}. \tag{7a}$$

The differences concern the value of C.

As has already been mentioned equation (6) deals with the growth of the bubble in the case of evaporation from the whole surface of the sphere. In our case, evaporation takes place from a part of the surface, that is from a layer of suspended liquid whose surface area is univocally determined by the opening angle β. Introducing this modification into equation (6) we get a form analogous to equation (7a) with a somewhat different coefficient C. From the energy balance of the expanding system, it can be shown that the opening angle over the whole range of evaporation is a constant. From the values of opening angle observed by Sideman [2], the error due to the above mentioned simplification does not exceed the range of variability of the constant C as obtained from different solutions of equation (6).

The rate of radius change is determined from equation (7)

$$\frac{\mathrm{d}R}{\mathrm{d}t} = N_{Ja}\left(\frac{3}{\pi t}\right)^{1/2} \tag{8}$$

Let us now discuss equation (2).

This equation is valid for the time range $t_0 - t$ where t_0 is the starting time of system and t is the time of complete evaporation.

The sum of expressions

$$\rho_v \cdot \tfrac{4}{3}\pi R^3 + m(t) = m_0$$

on the LHS of equation (2) is equal to the initial mass of the drop and, during evaporation, is a constant.

By introducing the quantity m_0 into equation (2) and transforming it we obtain:

$$\left(1+\frac{3m_0}{2\rho_L \pi R^3}\right)\frac{\mathrm{d}V}{\mathrm{d}t}+\frac{3}{R}\frac{\mathrm{d}R}{\mathrm{d}t}V$$
$$= 2g - \frac{2\rho_L g R}{1.82\sigma}V^2 - \frac{3g(\rho_{L_p}-\rho_L)}{2\rho_L \pi R^3}V_0$$
$$+\frac{3g(\rho_{L_p}-\rho_L)(1+\cos\beta)\varphi}{\rho_{L_p}\cdot\rho_L}\frac{t}{R}; \tag{9}$$

Since R is a time function [relationship (7) and equation (9)] it will be non-linear in the general case because of t, which makes its solution complicated. In order to avoid this difficulty let us examine the expression

$$\frac{3m_0}{2\rho_L \pi R^3} \quad \text{where} \quad m_0 = \rho_{L_p}\tfrac{4}{3}\pi R^{*3}$$

and R^* represents the initial radius of drop, then:

$$\frac{3m_0}{2\rho_L \pi R^3} = \frac{2\rho_{L_p}}{\rho_L}\left(\frac{R^*}{R}\right)^3. \tag{10}$$

The value of expression (10) for the liquids discussed and for an evaporation $\xi > 2\%$ is not bigger than 0.03. Neglecting the component

$$\frac{3m_0}{2\rho_L \pi R^3}$$

we are making an error of approximately 3% of the coefficient value at $\mathrm{d}V/\mathrm{d}t$. After this simplification the initial equation (2) will take the form:

$$\frac{\mathrm{d}V}{\mathrm{d}t}+\frac{3}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\cdot V$$
$$= 2g - \frac{2\rho_L\cdot g\cdot R}{1.82\sigma}V^2 - \frac{3g(\rho_{L_p}-\rho_L)V_0}{2\rho_L \pi R^3}$$
$$+\frac{3g(\rho_{L_p}-\rho_L)(1+\cos\beta)\cdot\varphi}{\rho_{L_p}\cdot\rho_L}\cdot\frac{t}{R}. \tag{11}$$

Incorporating into equation (11) the relationship (7), and transforming it, we obtain:

$$\frac{\mathrm{d}V}{\mathrm{d}t}+\tfrac{3}{2}Vt^{1/2}-2g+KV^2t^{1/2}+L\cdot t^{-3/2}+M\cdot t^{1/2}=0 \tag{12}$$

where:

$$K = \frac{4N_{Ja}}{1.82}\left(\frac{3\alpha}{\pi}\right)^{1/2}\frac{\rho_L g}{\sigma}$$

$$L = \frac{3}{2\pi\rho_L}(\rho_{L_p}-\rho_L)\frac{g v_0}{8N_{Ja}3(3\alpha/\pi)^{3/2}}$$

$$M = -\frac{3(1+\cos\beta)\cdot\varphi(\rho_{L_p}-\rho_L)}{4\rho_{L_p}\cdot\rho_L N_{Ja}(3\alpha/\pi)^{1/2}}.$$

Initial condition:

$$V|_{t=t_0} = N_{Ja}\left(\frac{3}{\pi t_0}\right)^{1/2}. \tag{13}$$

Equation (12) of the Ricatti type with the condition (13) is a univocally formulated differential problem. In the general case, however, it is not analytically solvable [11]. The problem was, therefore, solved using Merson's procedure with the help of an Odra 1204 computer. The obtained curve $V = V(t)$ is regular, and for times $t > 0.5$ s it is flat.

Since in direct experimental measurements one gets the curve of time-dependence of position, it would be more convenient to compare these data with the numerical solution of the form $H = H(t)$.

The curve is obtained by numerical differentiation of the solution of $V = V(t)$ in the form:

$$H(t) = H_0 + \int_{t_0}^{t} V(t)\,\mathrm{d}t \tag{14}$$

where H_0 is the position at the instant of separation from the capillary.

RESULTS AND DISCUSSION

Our experimental verification of theoretical results covered systems with different physico-chemical properties of the liquids examined for different superheating values of the evaporating liquid. The numerical solutions obtained were compared with experimental data given in [2] from where the results obtained for the evaporation of pentane in water for two different superheating values had been taken. The numerical solutions were also compared with the authors' own data obtained from experiments conducted using the apparatus whose diagram is presented in Fig. 2(a, b).

The liquid comprising the continuous phase fills a glass cylinder (Fig. 2a) to a certain level. The temperature of the liquid is equalized and stabilized by

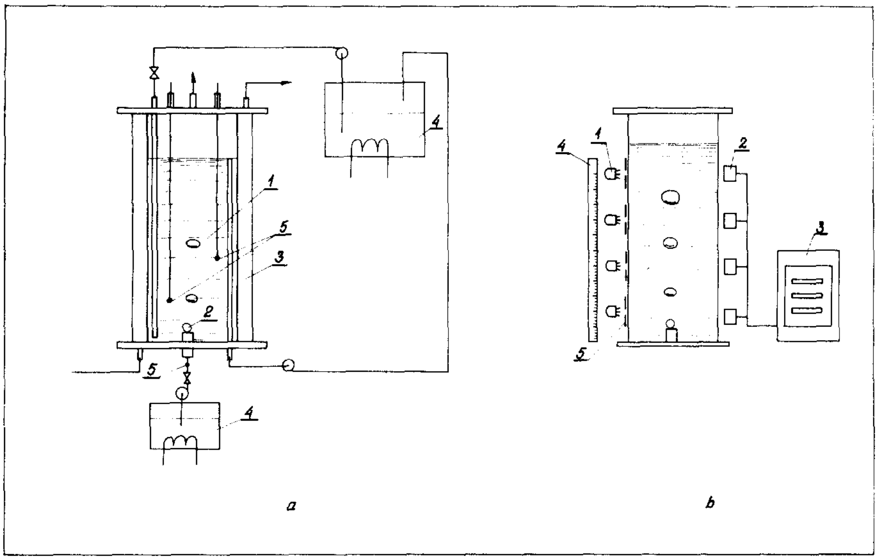

FIG. 2(a). Diagram of measuring apparatus: (1) continuous phase; (2) evaporating drop; (3) adiabatic jacket; (4) thermostat and (5) thermo-couples.

FIG. 2(b). Optical recording system: (1) source of light; (2) photo-tube; (3) time recorder; (4) measurement of light line position and (5) collimator of light.

means of the thermostat, 4. The heat losses into the surroundings are compensated for by applying a thermal jacket, 3. The temperature in the whole volume of the continuous phase is controlled using a thermocouple circuit, 3. To the continuous phase prepared in this way a drop of evaporating liquid pre-heated to boiling point is introduced. While vaporizing the drop is moving upwards. Its motion is recorded by an optical system (Fig. 2b). The optical system is composed of the pairs: light source–photo–tube connected to a time recorder, 3. The evaporating drop while crossing the light line interrupts the circuit thus stopping the time meter. The results were obtained in the form of the relationship: system position–time. In their own experiments the authors of this paper examined two systems: water–hexane and water–carbon tetrachloride each at two temperatures of superheating. Due to a difference in physicochemical properties (Table 1) of the phase forming the drop, the systems seem to be sufficiently representative for drawing conclusions as to the degree of generalization of the derived equation. The experimental classification is presented on the diagram of Fig. 3. The straight lines in the logarithmic coordinate system represent the numerical solutions; the points marked represent the results of experiments. The experimental time-dependence of position of the evaporating drop was approximated in the logarithmic

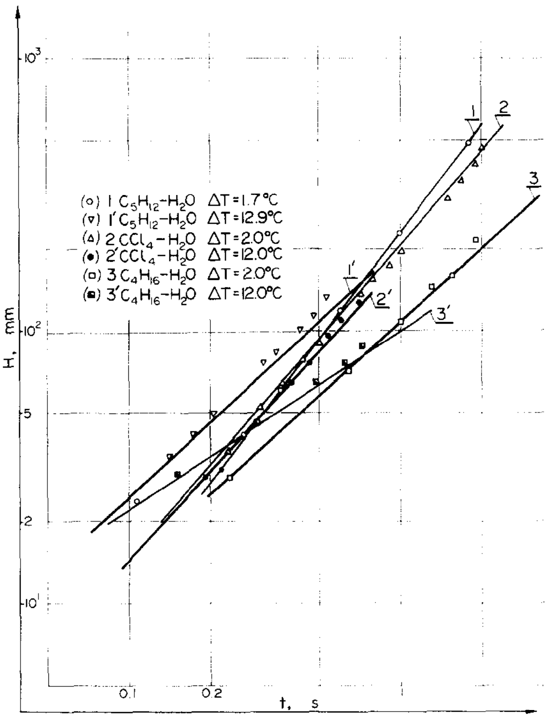

FIG. 3. Time-dependence of position of bubble-suspended liquid system.

Table 1. Physico-chemical data of the systems corresponding to the curves 1, 2 and 3

No.	C_p	ρ_v	ρ_L	ρ_{L_p}	α	V_0	T_∞	T_s	L
1	0.591	0.0031	0.9933	0.5763	$8.9 \cdot 10^{-4}$	$22.4 \cdot 10^{-3}$	310.9	309.3	85.3
2	0.208	0.0051	0.974	1.482	$7.4 \cdot 10^{-4}$	$7.3 \cdot 10^{-3}$	351.7	349.7	45.7
3	0.570	0.0028	0.9778	0.9590	$9.3 \cdot 10^{-4}$	$7.3 \cdot 10^{-3}$	344.1	342.1	80.5

Table 2

No.	1	1′	2	2′	3	3′
$P_{\text{expt.}}$	1.20	1.13	1.08	1.14	0.95	0.71
P_{theory}	1.32	1.09	1.15	1.10	0.89	0.65
$B_{\text{expt.}}$	195	288	196	205	115	108
B_{theory}	230	200	208	180	108	100

coordinate system with a straight line applying the method of least squares. The method of linear regression allows to establish that through the points determined by experiment one can draw a straight line. If we assume that the time-dependence of position of the evaporating drop may be represented by the expression (1)′ then in order to compare experiment with theory it is enough to check the values of coefficients B and p for the correlation lines and theoretical lines. Such a comparison is shown in Table 2. As can be seen from the comparison, the theoretical description of the motion of the evaporating drop using equation (12) is in good agreement with the real motion of the evaporating drop.

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EQUATION DU MOUVEMENT D'UNE GOUTTE DE VAPEUR EN EXPANSION AU SEIN D'UN LIQUIDE NON MISCIBLE

Résumé—En s'appuyant sur la description proposée par Sideman de l'évaporation interne d'une goutte avec formation d'une enveloppe liquide suspendue à la partie inférieure de la bulle en expansion, une équation est formulée qui décrit le mouvement d'un tel système. L'équation est résolue numériquement et les résultats comparés à la fois aux données expérimentales tirées de [2] et aux résultats expérimentaux obtenus par l'auteur. On trouve un bon accord entre théorie et expérience.

BEWEGUNGSGLEICHUNG FÜR EINE EXPANDIERENDE DAMPFBLASE IN EINER NICHT MISCHBAREN FLÜSSIGKEIT

Zusammenfassung—Unter Benützung der von Sideman vorgeschlagenen qualitativen Beschreibung des Mechanismus der inneren Verdunstung eines Tropfens mit gleichzeitiger Bildung einer im unteren Teil der expandierenden Blase aufgehängten Flüssigkeitshülle wurde eine Gleichung abgeleitet, die solch ein System beschreibt. Diese Gleichung wurde mittels numerischer Methoden gelöst. Die errechneten Ergebnisse wurden mit den experimentellen Daten aus der Arbeit [2] und eigenen verglichen.

Es wurde eine gute Übereinstimmung der errechneten und der experimentalen Ergebnissen festgestellt.

УРАВНЕНИЕ ДВИЖЕНИЯ РАСТУЩЕГО ПУЗЫРЬКА ПАРА В НЕСМЕШИВАЮЩЕЙСЯ ЖИДКОЙ СРЕДЕ

Аннотация — На основании качественного описания процесса испарения внутрь движущейся капли, данного Сайдеманом, выведено аналитическое уравнение движения растущего пузырька с плёнкой жидкости, расположенной в его нижней части.

Уравнение решено численными методами. Результаты расчётов сравниваются с экспериментальными данными Сайдемана [2] и авторов данной работы.

Получено хорошее соответствие между расчётными и опытными данными.